Analytic reconstruction of some dynamical systems

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Abstract

We propose a reconstruction of the initial system of ordinary differential equations from a single observed variable. The suggested approach is applied to a certain class of systems which includes, in particular, the Rössler system and other chaotic systems. We develop relations and a method to pass from a model that involves the observable and its time derivatives to a real original system. To this end, we first find a set of candidates of the system in an analytic way. After that, by additionally studying the system, we make a choice for the sought system.

Keywords: ordinary differential equations, observable, reconstruction, original system.

1. Introduction

The problem of reconstruction of a dynamical system from time series is topical in many fields of human activities. Various aspects and approaches to this problem have been treated in numerous studies. Most often, this problem is solved using numerical methods. For example, the authors in [1] have used a Legendre polynomial approximation of the vector field flow. The same flow method has been used in [2] to reconstruct maps or differential equations with hidden variables. Alternative algorithms have been proposed by many authors. In [3, 4, 5], the coefficients of the unknown system were sought for on the basis of a hierarchical approach. A two-stage Taylor series approach [6] has been used in [7] to construct an effective algorithm. Also, an improved algorithm is proposed in [8], where the optimization becomes more effective via a reduction of the initial parameter region. To achieve this, the author uses a cumulative backward differentiation formula. In order to improve the calculation algorithm, a perturbation method is used in [9] replacing the traditional Gauss-Newton or Levenberg–Marquardt methods. The mean square root method is used in [10] to find the coefficients in the Lorenz systems [11] and the Chen systems [12] for series with noises. H. Iba [13] has improved the mean square root method by using the genetic programming, which was also used in [14, 15]. A simplification of the calculation algorithm was proposed in [16] by reducing the number of the sought system parameters using a known relation between them. Good reconstruction results can be obtained via the Bock algorithm [17, 18].

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A probability approach to determine the coefficients of the equations, which generate the time series, has been used in [19], and for forecasting of the time series without constructing the model in [20]. In [21], the equation coefficients were determined by comparing the type of the attractor, the first return map, and the largest Lyapunov exponent that were obtained from the original system and the proposed model. A synchronization method was used for obtaining a model by using vector time series in [22]. A similar approach was used in [23, 24]. To forecast the time series, traditional reconstruction methods are widely used together with relatively new approaches, including neural networks [25], radial basis functions [26], fuzzy modeling [27], wavelet networks [28], etc.

The reconstruction problem we will be dealing with consists in a structure selection for an autonomous differential system with a polynomial right-hand side from a single observable variable. From this point of view, the approach proposed in [29, 30, 31, 32] is interesting. In [33], the following questions arising in the problem of global reconstruction were formulated: (1) Is it possible to create a model, within a certain class of models, that describes the initial time series? (2) Is the model unique? If not, what is the degree of nonuniqueness within the class of models treated? Answering these questions in this reference, numerical methods were important, in particular, the genetic algorithm was used.

In this paper, we will show that the structure of the model can be obtained for a certain class of systems in terms of mainly strict mathematical transformations with a minimal use of computational operations. In addition to the structure, this approach allows also to obtain relations between the coefficients of the sought equations, which brings the researcher closer to an ideal solution of the reconstruction problem, which is to find an exact model for the original system with a unique true choice for the coefficients. A procedure to solve the reconstruction problem could be as follows.

1. Using a numerical procedure, construct a standard system [30] using the observable and its derivatives.
2. Using analytic relations between the coefficients of the standard and the original systems obtain a family of candidate systems [37] containing the original system.
3. Using additional information (if this information suffices) choose a unique real original system from the family of candidate systems.
4. If the additional information does not allow to make the choice, conduct an additional investigation. It is assumed that conditions of the experiment allow for conducting such an investigation.

It can be seen from the given procedure that it contains only one typical numerical operation (item 1) that can be repeated if necessary.

The paper is organized as follows. In Section 2, we consider a relatively simple class of systems that could have chaotic dynamics. For such a class, we deduce relations that could be used for reconstructing the unknown system from a single observable. Section 3.1 contains examples where we apply these relations to find analogs of the Rössler system that would have the same observable as the original system. Sections 3.2, 3.3 deal with finding, in a given class, all systems that can be the original system (items 1 and 2 of the above

procedure). In Section 3.4, we discuss choosing the original system from the set of candidate systems (items 3 and 4 from the above procedure).

2. Essentials of the approach

Consider a system

$$\dot{x}_i = X_i(x_1, \ldots, x_i, \ldots, x_n), \tag{1}$$

where the variables x_i, $i = 1, \ldots, n$, define the state of the process, X_i are polynomial functions. Following [30], we will call system (1) the Original System (OS). Assume we know one solution of (1), a function $x_i(t)$. Since, without any further information about system (1), it can not be recovered from data about only one observable $x_i(t)$, system (1) is often replaced with a Standard System (SS),

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3, \\ \quad \cdots \\ \dot{y}_n = Y(y_1, \ldots, y_n), \end{cases} \tag{2}$$

where $y_1(t) \equiv x_i(t)$, $Y(y_1, \ldots, y_n)$ is often taken to be a polynomial or a rational function. Here, many properties of systems (1) and (2) are the same.

It is clear that, in a general case, such a model can not give an exact realization of the physics of the process the way the OS does. Also, the function $y_1(t)$ obtained when system (2) is reconstructed could be approximately the same as the function $x_i(t)$, $y_1(t) \approx x_i(t)$. At the same time, there are particular cases where $x_i(t)$ and $y_1(t)$ coincide, that is, there is a system of type (2) such that $y_1(t)$, being its solution, satisfies the identity

$$y_1(t) \equiv x_i(t). \tag{3}$$

Such an exact substitution can be performed if there exists an inverse standard transformation (IST) that connects the variables in (1) and (2) [29].

It is clear that numerical methods can be applied to a wider class of systems, however, it may be useful sometimes to use an analytic approach for finding the OS. The complexity of this approach depends, among other things, on the general form of the OS, namely, on the number of the variables and the form of the polynomials that enter the right-hand side of system (1). For example, even if the system contains three variables and the degree of the polynomials is two, the general form of system (1) can be fairly complex,

$$\begin{cases} \dot{x}_1 = a_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_1^2 + a_5x_1x_2 \\ \qquad + a_6x_1x_3 + a_7x_2^2 + a_8x_2x_3 + a_9x_3^2, \\ \dot{x}_2 = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_1^2 + b_5x_1x_2 \\ \qquad + b_6x_1x_3 + b_7x_2^2 + b_8x_2x_3 + b_9x_3^2, \\ \dot{x}_3 = c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_1^2 + c_5x_1x_2 \\ \qquad + c_6x_1x_3 + c_7x_2^2 + c_8x_2x_3 + c_9x_3^2. \end{cases} \tag{4}$$

Correspondingly, if the transformations that connect systems (4) and (2) exist, they will also be complex. It was suggested in [34, 35] to use the ansatz library that permits to obtain various versions of the OS from relations between the coefficients of the OS and the SS.

Sometimes, it may happen that a model for complex processes, including chaotic processes, can have a rather simple form. Although the general system (4) contains 18 nonlinear terms, the Lorenz system [11], for example, contains only 2 such terms,

$$\begin{cases} \dot{x}_1 = -\sigma(x_1 - x_2), \\ \dot{x}_2 = -x_1 x_3 - x_2 + \rho x_1, \\ \dot{x}_3 = -\beta x_3 + x_1 x_2, \end{cases} \tag{5}$$

where σ, ρ, β are constant parameters. The Rössler system [36],

$$\begin{cases} \dot{x}_1 = -x_2 - x_3, \\ \dot{x}_2 = x_1 + a x_2, \\ \dot{x}_3 = b - c x_3 + x_1 x_3, \end{cases} \tag{6}$$

where a, b, c are parameters, has even simpler form. System (6) has only one nonlinear term which is a product of two different variables. As opposed to the general form of system (4), systems (5) and (6) do not contain squares of the variables. Having this in mind, one can consider replacing the general type system (4) with a simplified system of the form

$$\begin{cases} \dot{x}_1 = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3, \\ \dot{x}_2 = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3, \\ \dot{x}_3 = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1 x_2 + c_5 x_1 x_3 + c_6 x_2 x_3. \end{cases} \tag{7}$$

Definition 1. We say that a system belongs to the Rössler class (R-class) if it is of the form (7) and only one of the coefficients c_4, c_5, c_6 is distinct from zero.

Choose the variable x_2 to be the observable in the Rössler system, that is, $y_1 = x_2$. Its reconstruction, by [32], can be obtained in the form

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3 \\ \dot{y}_3 = A_0 + A_1 y_1 + A_2 y_2 + A_3 y_3 + A_4 y_1^2 + A_5 y_1 y_2 + A_6 y_1 y_3 \\ \qquad + A_7 y_2^2 + A_8 y_2 y_3 + A_9 y_3^2, \end{cases} \tag{8}$$

where $A_0, \dots, A_9$ are constants with

$$A_9 = 0. \tag{9}$$

It is easy to show that relation (9) holds not only for the Rössler system but for systems of a more general type.

Definition 2. We say that a system belongs to a shortened Rössler class (SR-class), if it is of the R-class and $b_3 = 0$, that is, a general SR-system is of the form

$$\begin{cases} \dot{x}_1 = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3, \\ \dot{x}_2 = b_0 + b_1 x_1 + b_2 x_2, \\ \dot{x}_3 = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1 x_2 + c_5 x_1 x_3 + c_6 x_2 x_3. \end{cases} \tag{10}$$

Proposition 1. *If the OS has form (10), then its reconstruction to a SS will have the form*

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3, \\ \dot{y}_3 = A_0 + A_1 y_1 + A_2 y_2 + A_3 y_3 + A_4 y_1^2 + A_5 y_1 y_2 + A_6 y_1 y_3 \\ \qquad + A_7 y_2^2 + A_8 y_2 y_3 \end{cases} \tag{11}$$

and the relations between the coefficients in systems (10) and (11) is given by

$$\begin{cases} A_0 = a_3 \begin{vmatrix} c_0 & c_1 \\ b_0 & b_1 \end{vmatrix} + \dfrac{1}{b_1} \begin{vmatrix} c_3 & c_5 \\ b_0 & b_1 \end{vmatrix} \begin{vmatrix} b_0 & b_1 \\ a_0 & a_1 \end{vmatrix}, \\ A_1 = a_3 \left(\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} - b_0 c_4 \right) \\ \qquad + \dfrac{1}{b_1} \left(\begin{vmatrix} c_3 & c_5 \\ b_0 & b_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 \\ c_5 & c_6 \end{vmatrix} \begin{vmatrix} b_0 & b_1 \\ a_0 & a_1 \end{vmatrix} \right), \\ A_2 = \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} + a_3 c_1 + \dfrac{c_5}{b_1} \begin{vmatrix} b_0 & b_1 \\ a_0 & a_1 \end{vmatrix} + \dfrac{a_1 + b_2}{b_1} \begin{vmatrix} b_0 & b_1 \\ c_3 & c_5 \end{vmatrix}, \\ A_3 = a_1 + b_2 + \dfrac{1}{b_1} \begin{vmatrix} c_3 & c_5 \\ b_0 & b_1 \end{vmatrix}, \\ A_4 = \dfrac{1}{b_1} \begin{vmatrix} b_1 & b_2 \\ c_5 & c_6 \end{vmatrix} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} - a_3 b_2 c_4, \\ A_5 = a_3 c_4 + \dfrac{c_5}{b_1} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \dfrac{a_1 + b_2}{b_1} \begin{vmatrix} c_5 & c_6 \\ b_1 & b_2 \end{vmatrix}, \\ A_6 = \dfrac{1}{b_1} \begin{vmatrix} b_1 & b_2 \\ c_5 & c_6 \end{vmatrix}, \\ A_7 = -\dfrac{c_5}{b_1} (a_1 + b_2), \\ A_8 = \dfrac{c_5}{b_1}, \\ b_1 \neq 0. \end{cases} \tag{12}$$

A proof of Proposition 1 is given in Appendix A.

Hence, one can infer that if the reconstruction of the unknown OS from one observable is of form (11), then the OS can be of the SR-class. This fact can be used in the analytical search for the form of the OS, replacing the calculation procedure.

The Rössler system (6), with the notations as in (10), can be represented as

$$\begin{cases} \dot{x}_1 = a_2x_2 + a_3x_3, \\ \dot{x}_2 = b_1x_1 + b_2x_2, \\ \dot{x}_3 = c_0 + c_3x_3 + c_5x_1x_3, \end{cases} \tag{13}$$

where $a_2 = -1$, $a_3 = -1$, $b_1 = 1$, $b_2 = a$, $c_0 = b$, $c_3 = -c$, $c_5 = 1$. Since $a_0 = a_1 = b_0 = c_1 = c_2 = c_4 = c_6 = 0$ in this case, the expressions in (12) for the Rössler system become

$$\begin{cases} A_0 = a_3b_1c_0 = -b, & A_1 = -a_2b_1c_3 = -c, \\ A_2 = a_2b_1 - b_2c_3 = -1 + ac, & A_3 = b_2 + c_3 = a - c, \\ A_4 = a_2b_2c_5 = -a, & A_5 = c_5(b_2^2/b_1 - a_2) = a^2 + 1, \\ A_6 = A_7 = -b_2c_5/b_1 = -a, & A_8 = c_5/b_1 = 1. \end{cases} \tag{14}$$

Using relations (14) one can obtain systems of type (11) that would have the solution $y_1(t)$ coinciding with a solution $x_2(t)$ of the original Rössler system.

Let us state an obvious general result that we will use in the following.

Proposition 2. *If two different OS's have the same solution considered as the observable $y_1(t)$, then the SS reconstructed from this observable will be the same for both OS's.*

3. Results

3.1. Analogs of the Rössler system

Using Proposition 2 and relations (14) one can obtain different versions of the OS, which would have the same structure, that is, the nonzero coefficients in the right-hand sides will be the same as in the Rössler system. The values of the coefficients of the new system will differ from those in the Rössler system but the function $x_2(t)$ will coincide with the function $x_2(t)$ in the Rössler system.

As an example, consider a reconstruction of the Rössler system for $a = 0.15$, $b = 0.2$, $c = 10$. The functions $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_2(x_1)$ are presented for this case in Fig. 1. For system (13), we get $a_2 = a_3 = -1, b_1 = 1, b_2 = 0.15, c_0 = 0.2, c_3 = -10, c_5 = 1$. By (14), $A_0 = a_3b_1c_0$. If, for example, b_1 is unchanged, and the coefficient c_0 is increased by 10 times, then for A_0 to be unchanged, the absolute value of a_3 must be reduced also by 10 times. New values of the coefficients will be $c_0 = 2, a_3 = -0.1$. Since these changes do not change the value of A_0 and the coefficients $A_1, \ldots A_8$ do not depend on c_0 and a_3, hence remain unchanged, Proposition 2 can be applied. Hence, the solution $x_2(t)$ of the new OS must coincide with $x_2(t)$ of the original Rössler system. This is illustrated in Fig. 2. The graphs show that the functions $x_1(t)$ and $x_2(t)$ remain unchanged, see Fig. 1, and, for the function x_3, the scale has changed. The scaling problem in more detail has been considered in [33].

Using relations (12) it is also possible to obtain a modification of the Rössler system with a change of the structure, for example, like the following:

$$\begin{cases} \dot{x}_1 = a_2x_2 + a_3x_3, \\ \dot{x}_2 = b_0 + b_1x_1 + b_2x_2, \\ \dot{x}_3 = c_0 + c_5x_1x_3. \end{cases} \tag{15}$$

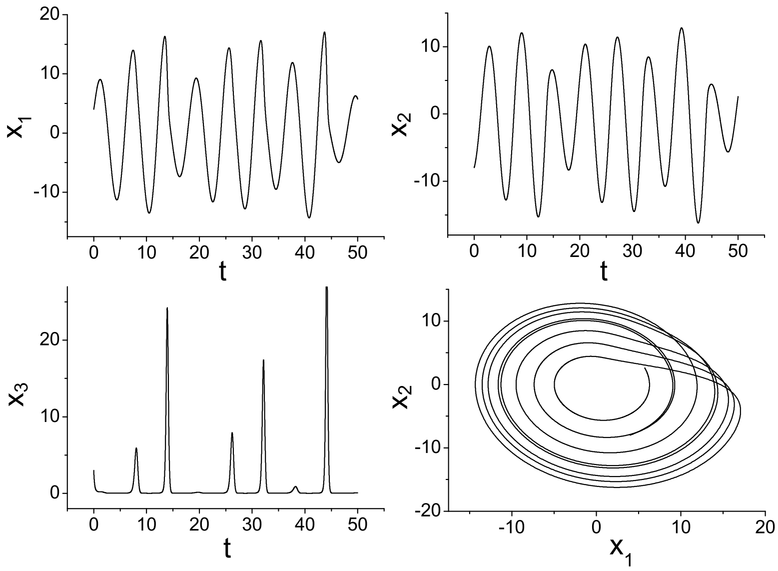

Figure 1: The functions $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_2(x_1)$ in the original Rössler system (13). Numeric integration is carried out with the fourth-order Runge-Kutta method on the time interval of 50 sec. with step 0.005 sec.

Comparing systems (15) and (13), one can see that in system (15) $c_3 = 0$ but $b_0 \neq 0$. Then, for system (15), we get from (12) that

$$\begin{cases} A_0 = a_3 b_1 c_0, & A_1 = a_2 b_0 c_5, \\ A_2 = a_2 b_1 + b_0 b_2 c_5 / b_1, & A_3 = b_2 - b_0 c_5 / b_1, \\ A_4 = a_2 b_2 c_5, & A_5 = c_5 (b_2^2 / b_1 - a_2), \\ A_6 = A_7 = -b_2 c_5 / b_1, & A_8 = c_5 / b_1. \end{cases} \tag{16}$$

The analysis given in Appendix B shows that by setting $b_0 = c$ and leaving a_2, a_3, b_1, b_2, c_0, c_5 as in system (13), the values of the coefficients in the reconstruction (16) will be the same as the coefficients in the reconstruction (14). Consequently, system (15) satisfies the conditions in Proposition 2. Hence the solution $x_2(t)$ of system (15) will coincide with the solution $x_2(t)$ of system (13) for an appropriate choice of the initial conditions. This is illustrated in Fig. 3, where one can also see that the function $x_3(t)$ coincides with that for the Rössler system, and $x_1(t)$ is shifted as compared with $x_1(t)$ for system (13).

3.2. Finding the OS from its reconstruction in the form of a SS

In some cases, the proposed approach could be effective when recovering an unknown model from its reconstruction (11). As an example, consider the system

$$\begin{cases} \dot{x}_1 = a_2 x_2 + a_3 x_3, \\ \dot{x}_2 = b_1 x_1, \\ \dot{x}_3 = c_1 x_1 + c_3 x_3 + c_4 x_1 x_2, \end{cases} \tag{17}$$

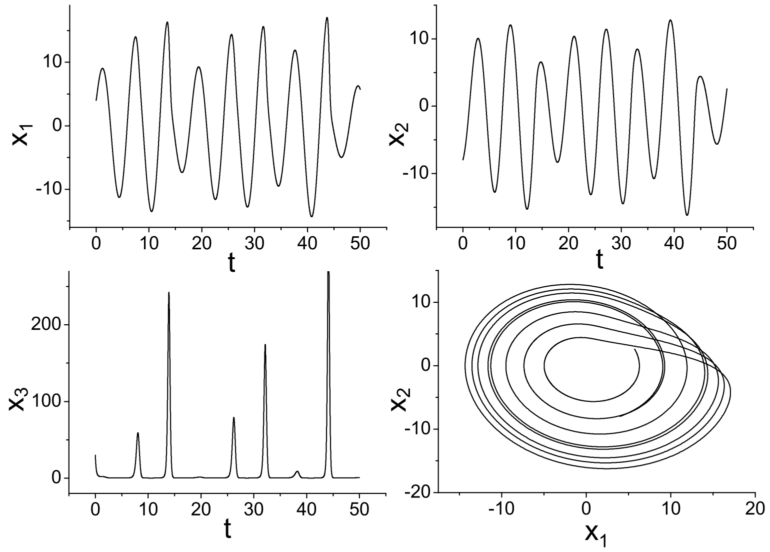

Figure 2: The same as in Fig. 1 for the Rössler system with the values of c_0 and a_3 changed.

where $a_2 = -25.4, a_3 = 1, b_1 = 1, c_1 = 15.4, c_3 = -10, c_4 = -143$. This and all subsequent systems were solved by applying the fourth-order Runge-Kutta method on the time interval of 25 sec. with step 0.001 sec. The obtained time series for $x_2(t)$ in system (17) was taken to be a unique observable, that is, we took $y_1(t) \equiv x_2(t)$ for the reconstruction in the form of the SS. The reconstructed SS of the form (11) has the coefficients shown in Table 1. The reconstruction was obtained by applying both a numerical method, similar to [29], and an analytical method using relations (12). As Table 1 shows, the errors in the values of the coefficients in the reconstruction obtained numerically are small. In the following, when making necessary transformations we used exact values obtained analytically and shown in the right column of the table. As a result, the SS has the form

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3, \\ \dot{y}_3 = A_1 y_1 + A_2 y_2 + A_3 y_3 + A_5 y_1 y_2, \end{cases} \tag{18}$$

where $A_1 = -254, A_2 = -10, A_3 = -10, A_5 = -143$. Solutions of system (18) and its phase portraits are shown in Fig. 4.

We will consider the OS (17) to be unknown. To obtain information about it, we use (12) and (18). The results of the analysis given in Appendix C show that the sought OS has $a_0 = b_2 = c_0 = c_5 = c_6 = 0$ and the nonzero coefficients could only be the following:

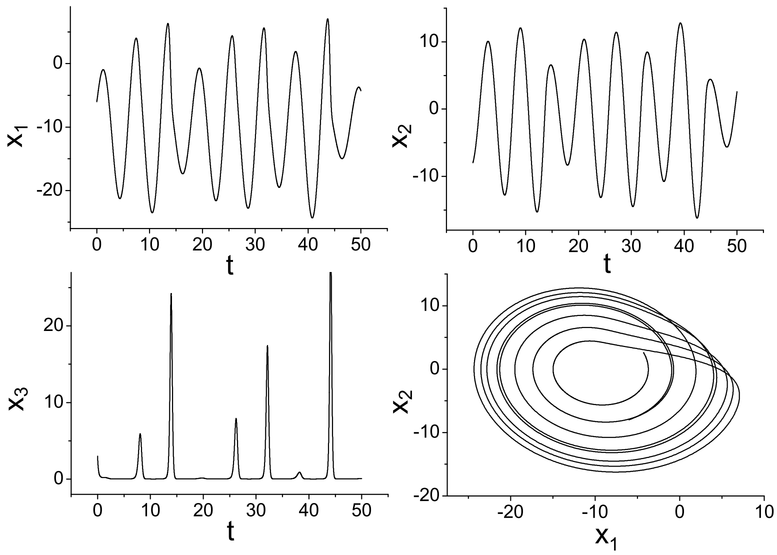

Figure 3: The same as in Fig. 1 for system (15).

$a_1, a_2, a_3, b_0, b_1, c_1, c_2, c_3, c_4$. This significantly simplifies system (12) giving

$$\begin{cases} A_0 = a_1 b_0 c_3 - a_3 b_0 c_1, \\ A_1 = a_3 b_1 c_2 - a_2 b_1 c_3 - a_3 b_0 c_4, \\ A_2 = a_2 b_1 + a_3 c_1 - a_1 c_3, \\ A_3 = a_1 + c_3, \\ A_5 = a_3 c_4, \\ A_4 = A_6 = A_7 = A_8 = 0. \end{cases} \tag{19}$$

To pass to the reconstruction of the OS, it is necessary to determine how many of the 9 nonzero coefficients will be contained in the sought equations and which of them. As a simplest version of the OS, one can take the following system that has the same structure as system (18):

$$\begin{cases} \dot{x}_1 = a_3 x_3, \\ \dot{x}_2 = b_1 x_1, \\ \dot{x}_3 = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1 x_2. \end{cases} \tag{20}$$

The quantities in systems (18) and (20) are connected with the relation $x_2 = y_1$, $a_3 = 1$, $b_1 = 1$, $c_1 = A_2$, $c_2 = A_1$, $c_3 = A_3$, $c_4 = A_5$.

It is clear that system (20) is not the only possible one for the OS recovered from the SS (18). Denote by K the total number of nonzero coefficients in the right-hand sides of the OS. We assume, similarly to (20), that $K = 6$ for other unknown OS's (the case where

Coefficients of SS	Numerical method	Analytical method
A_0	$-3.1427 \cdot 10^{-5}$	0
A_1	-253.9956	-254
A_2	-9.9995	-10
A_3	-9.9998	-10
A_4	$6.6653 \cdot 10^{-5}$	0
A_5	-142.9986	-143
A_6	$-2.3495 \cdot 10^{-4}$	0
A_7	$1.2581 \cdot 10^{-4}$	0
A_8	$-2.4444 \cdot 10^{-4}$	0
A_9	$-1.1524 \cdot 10^{-5}$	0

Table 1: Values of the coefficients in reconstruction (11) for the solution $x_2(t)$ of the OS (17) obtained numerically and analytically using relation (12). The value of the coefficient A_9 obtained by numerical calculations verifies relation (9).

$K \neq 6$ will be considered in Section 3.3). Since relations (19) involve 9 coefficients, when considering candidate systems [37] to choose a unique real OS from, it is necessary to check relations (19) for possible combinations of 6 coefficients from 9 as to satisfy conditions in Proposition 2. To this end, it is sufficient to alternatively set 3 out of 9 coefficients to zero and check whether the structure of the SS is that of (18). Such transformations are carried out in Appendix C. As the result we get the OS's $S_1 - S_8$, see Table 2.

The functions $x_2(t)$ for all systems $S_1 - S_8$ are practically coincide with the time series for $y_1(t)$ used in the reconstruction, see Fig. 4. Insignificant differences are due to numerical

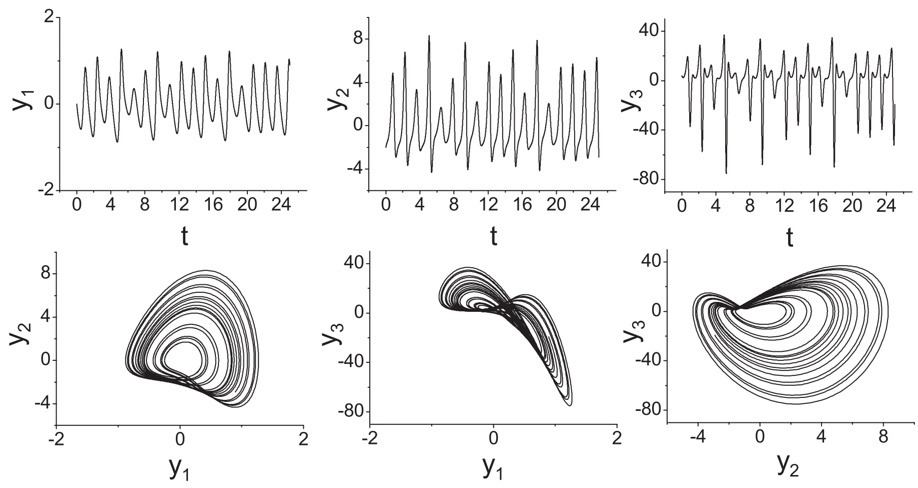

Figure 4: The functions and phase portraits for system (18); $y_1(t)$ is the observable $x_2(t)$ in system (17).

integration errors and rounding the fractional values of the coefficients obtained from (19) for some OS's. The phase curves for systems $S_2 - S_8$ are shown in Fig. 5. The phase

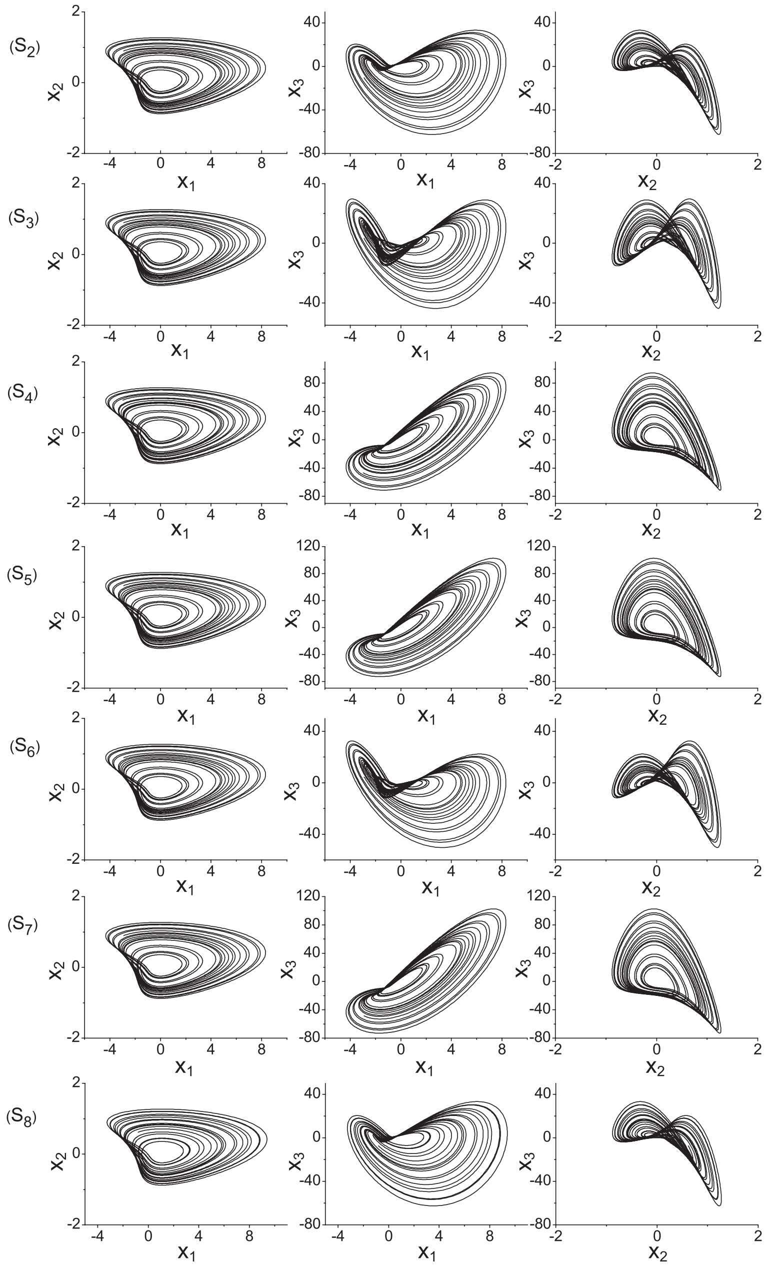

Figure 5: Phase portraits of systems $S_2 - S_8$.

K	OS	Coefficients of OS								
		a_1	a_2	a_3	b_0	b_1	c_1	c_2	c_3	c_4
6	S_1			$\times$		$\times$	$\times$	$\times$	$\times$	$\times$
	S_2		$\times$	$\times$		$\times$		$\times$	$\times$	$\times$
	S_3		$\times$	$\times$		$\times$	$\times$		$\times$	$\times$
	S_4	$\times$		$\times$		$\times$		$\times$	$\times$	$\times$
	S_5	$\times$		$\times$		$\times$	$\times$	$\times$		$\times$
	S_6	$\times$	$\times$	$\times$		$\times$			$\times$	$\times$
	S_7	$\times$	$\times$	$\times$		$\times$		$\times$		$\times$
	S_8		$\times$	$\times$	$\times$	$\times$			$\times$	$\times$
7	S_9		$\times$	$\times$		$\times$	$\times$	$\times$	$\times$	$\times$
	S_{10}		$\times$	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$
	S_{11}	$\times$		$\times$		$\times$	$\times$	$\times$	$\times$	$\times$
	S_{12}	$\times$	$\times$	$\times$		$\times$		$\times$	$\times$	$\times$
	S_{13}	$\times$	$\times$	$\times$		$\times$	$\times$		$\times$	$\times$
	S_{14}	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$		$\times$
	S_{15}	$\times$	$\times$	$\times$	$\times$	$\times$		$\times$		$\times$
8	S_{16}	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$	$\times$
9	S_{17}	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

Table 2: OS's corresponding to reconstruction (18): K is the number of the nonzero coefficients in the right-hand sides of the OS's. System (17) is denoted by S_3, system (20) by S_1, and system (24) by S_{17}.

trajectories of system S_1 are similar to the trajectories of system (18). Relations for the coefficients for the SS and all OS's obtained from (19) are shown in Table 3.

3.3. Determining a complete family of candidate systems

In the previous section, we have considered 8 possible OS's that have 6 coefficients in the right-hand sides. Using relation (19) one can find the number K of possible nonzero coefficients in the OS that has system (18) as a SS. First, let us find a lower bound for K, denoted by $K_{\min}$.

Definition 3. An OS will be called minimal for the SS (18) if it permits a reconstruction in the form of (18) and has $K_{\min}$ nonzero coefficients, and any system of the form (10) with $K < K_{\min}$ can not give a reconstruction in the form of (18).

Let us check whether systems $S_1 - S_8$ are minimal, that is, whether there exists an OS with $K < 6$. To do this, it is necessary to equate various coefficients in these systems to zero, but the coefficients A_1, A_2, A_3, A_5 in the SS should be distinct from zero. It turned out that there is a unique system with $K = 5$ that satisfies this condition,

$$\begin{cases} \dot{x}_1 = a_2 x_2 + a_3 x_3, \\ \dot{x}_2 = b_1 x_1, \\ \dot{x}_3 = c_3 x_3 + c_4 x_1 x_2. \end{cases} \tag{21}$$

K	OS	Relations between the coefficients of SS and OS			
		A_1	A_2	A_3	A_5
6	S_1	$a_3b_1c_2$	a_3c_1	c_3	a_3c_4
	S_2	$a_3b_1c_2 - a_2b_1c_3$	a_2b_1	c_3	a_3c_4
	S_3	$-a_2b_1c_3$	$a_2b_1 + a_3c_1$	c_3	a_3c_4
	S_4	$a_3b_1c_2$	$-a_1c_3$	$a_1 + c_3$	a_3c_4
	S_5	$a_3b_1c_2$	a_3c_1	a_1	a_3c_4
	S_6	$-a_2b_1c_3$	$a_2b_1 - a_1c_3$	$a_1 + c_3$	a_3c_4
	S_7	$a_3b_1c_2$	a_2b_1	a_1	a_3c_4
	S_8	$-a_2b_1c_3 - a_3b_0c_4$	a_2b_1	c_3	a_3c_4
7	S_9	$a_3b_1c_2 - a_2b_1c_3$	$a_2b_1 + a_3c_1$	c_3	a_3c_4
	S_{10}	$a_3b_1c_2 - a_2b_1c_3 - a_3b_0c_4$	a_2b_1	c_3	a_3c_4
	S_{11}	$a_3b_1c_2$	$a_3c_1 - a_1c_3$	$a_1 + c_3$	a_3c_4
	S_{12}	$a_3b_1c_2 - a_2b_1c_3$	$a_2b_1 - a_1c_3$	$a_1 + c_3$	a_3c_4
	S_{13}	$-a_2b_1c_3$	$a_2b_1 + a_3c_1 - a_1c_3$	$a_1 + c_3$	a_3c_4
	S_{14}	$a_3b_1c_2$	$a_2b_1 + a_3c_1$	a_1	a_3c_4
	S_{15}	$a_3b_1c_2 - a_3b_0c_4$	a_2b_1	a_1	a_3c_4
8	S_{16}	$a_3b_1c_2 - a_2b_1c_3$	$a_2b_1 + a_3c_1 - a_1c_3$	$a_1 + c_3$	a_3c_4
9	S_{17}	$a_3b_1c_2 - a_2b_1c_3 - a_3b_0c_4$	$a_2b_1 + a_3c_1 - a_1c_3$	$a_1 + c_3$	a_3c_4

Table 3: Relations between the coefficients of the SS (18) and OS's corresponding to (19).

Here, by (19),

$$A_1 = -a_2b_1c_3, \quad A_2 = a_2b_1, \quad A_3 = c_3, \quad A_5 = a_3c_4. \tag{22}$$

Although A_1, A_2, A_3, A_5 are nonzero, it follows from (22) that

$$A_1 = -A_2A_3. \tag{23}$$

By substituting the values of A_1, A_2, A_3 from Table 1 into (23), we see that (23) is not true. Consequently, relations (22) do not permit to obtain a reconstruction of (18), and system (21) can not be an OS. So, the minimal OS's could only be those with $K = 6$ in Table 2.

The analysis in Appendix C shows that the maximal value of K so that the corresponding OS is reconstructed from (18) is $K_{\max} = 9$. That is, a maximal OS will contain all the coefficients that enter the right-hand sides of relations (19). It has the form

$$\begin{cases} \dot{x}_1 = a_1x_1 + a_2x_2 + a_3x_3, \\ \dot{x}_2 = b_0 + b_1x_1, \\ \dot{x}_3 = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_1x_2. \end{cases} \tag{24}$$

Since $K = 9$ for this system and the number of known coefficients in the SS (18) is 4, some values for the coefficients $a_1, \ldots, c_4$ were taken arbitrarily ($a_3 = b_1 = 1$) whereas the others

were determined from (19). As the result we get $a_1 = -5$, $a_2 = -10$, $b_0 = -1$, $c_1 = 25$, $c_2 = -61$, $c_3 = -5$, $c_4 = -143$.

Now, to find all possible OS's other than with $K_{\min} = 6$ and $K_{\max} = 9$, consider possible OS's with $K = 7$ and $K = 8$. We can previously estimate the possible number of OS's for each value of K. Three of the coefficients, a_3, b_1, and c_4, are always nonzero (see Appendix C), whereas the other 6 may or may not be zero. If $K = 8$, the system must have 5 more nonzero coefficients. Various cases can be obtained by in turn equating one of the six unknown coefficients to zero. The number of such combinations will be $n_8 = \binom{6}{1} = 6$. Similarly, if $K = 7$, we get $n_7 = \binom{6}{2} = 15$, and for $K = 6$, $n_6 = \binom{6}{3} = 20$. Each combination of zero and nonzero coefficients must be checked with relations (19). If the reconstruction of an OS of the form (18) is impossible for some combination of the coefficients, then this combination is rejected. Such an analysis has been carried out in Section 3.2 and Appendix C for $K = 6$. As is indicated, see Table 2, the number of possible OS's in this case is 8 which is much less than $n_6 = 20$.

A similar analysis has been done for $K = 7$ and $K = 8$. The results are shown in Tables 2 and 3. Hence, the total number of possible OS's is 17.

3.4. Choosing an OS from candidate systems

In order to choose an OS from candidate systems given in Table 2, one can use an additional information about the system. For example, if we know that the variable x_1 depends on x_2 or x_3, then the OS must contain the coefficients a_2 or a_3, correspondingly. If we know that the change of the variable x_3 depends on its value, then the OS must have the coefficient c_3, etc. In other words, we need to find features of the OS that single it out from other candidate systems. Systems with no such features should be excluded from the consideration.

There can arise situations where additional information regarding the OS is not sufficient or is missing at all. The missing features of the OS can be obtained in additional experiments if the conditions of the experiment allow for a change of the parameters, i.e., the coefficients of the OS. Namely, if the conditions of the experiment change at least one of the coefficients of the OS, then the observable $y_1(t) \equiv x_2(t)$ will change. The reconstruction in form (18) obtained from this observable will have the value of at least one of the coefficients $A_1, \ldots, A_5$ changed. By analyzing these changes, one can formulate new features of the sought OS.

The study can be carried out along the following scheme, see Fig. 6.

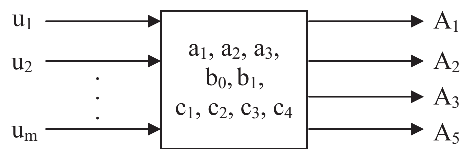

Figure 6: The scheme for determining the structure of the OS.

1. With some process conditions $(u_{11}, ..., u_{m1})$, we get a time series $y_{11}(t)$ for the observable $y_1(t)$ that is used for the reconstruction of type (18) with some coefficients $A_{11}, A_{21}, A_{31}, A_{51}$.
2. For new process conditions $(u_{12}, \ldots, u_{m2})$ we get a new time series $y_{12}(t)$ and new reconstruction of type (18) with new coefficients $A_{12}, A_{22}, A_{32}, A_{52}$.
3. Repeat step 2 by trying to change other parameters if the experiment conditions permit to do so. For example, if step 2 has been performed with the process temperature changes, then we repeat it with the pressure changes, etc.
4. By performing step 3 with various process conditions, we determine which of the coefficients in the SS change.
5. Using the results of the experiments we formulate features for choosing the initial system.

For making the analysis easier, one can use Table 4 obtained from Table 3. It clearly shows which of the coefficients of various OS's determine the values of the coefficients in the SS. For example, according to Table 4, if only the coefficient A_1 is changed under some actions in the SS (18), this could have occurred due to a change of b_0 or b_1 or c_2. And if a coefficient of the OS changes, this means that this coefficient is not zero. Consequently, $b_0 \neq 0$ or $b_1 \neq 0$ or $c_2 \neq 0$ in the OS. Similarly, if only A_2 in the reconstructed system is changed when different actions were applied, then this would mean that $a_2 \neq 0$ or $c_1 \neq 0$ in the OS. If only A_3 in the SS is changing, then one can assume that either $a_1 \neq 0$ or $c_3 \neq 0$ in the OS.

It is also convenient for making the analysis to use Table 5 obtained from Table 4. It shows which coefficients in the SS may change when different actions are applied to the OS. The information about the change of the coefficients of the OS is not used in this case. When constructing Table 5, various cases that may occur when external effects are applied to the OS were considered. For example, if the coefficient A_1 changes when using reconstruction (18) from the observable in system S_2, then, according to Table 4, this can occur if one of the coefficients of the OS changes, namely the coefficient c_2. If system S_1 is considered, a change of A_1 can only occur if b_1 or c_2 change in the OS. All these cases are reflected in Table 5.

A similar approach is used for combinations of A's. For example, if system S_2 is reconstructed, then simultaneous change of the coefficients A_1 and A_3 is possible if the only coefficient c_3 in the OS is changed. But when reconstructing system S_7, a change of A_3 can occur if a_1 is changed, whereas changes A_1 if so does c_2. But if a change of the conditions of the experiment implies simultaneous change of a_1 and c_2 in the OS, that would imply that both A_1 and A_3 change.

Also, it is impossible for system S_{12} that a change of only one of the coefficients in the OS would cause simultaneous change of the coefficients A_1, A_2, A_5 in the SS (and A_3 remains the same). But a change of a_2 would imply changes of A_1 and A_2, and a change of a_3 would imply the same for A_1 and A_5. If this occurs for the same external effect, we get a number of changed coefficients A_1, A_2, A_5, which is indicated in Table 5.

Combinations given in Table 5 can be used as features for choosing the OS. For example, if the SS shows a change of only coefficient A_2, then, according to Table 5, this can happen only if the OS's S_1, S_3, S_5, S_7, S_9, S_{11}, S_{13}, S_{14}, S_{15}, S_{16}, and S_{17} are considered, and this can not happen for the OS's S_2, S_4, S_6, S_8, S_{10}, S_{12}. An example of choosing the OS is given in Appendix D.

OS	Coefficients of OS								
	a_1	a_2	a_3	b_0	b_1	c_1	c_2	c_3	c_4
S_1	—	—	A_1, A_2, A_5	—	A_1	A_2	A_1	A_3	A_5
S_2	—	A_1, A_2	A_1, A_5	—	A_1, A_2	—	A_1	A_1, A_3	A_5
S_3	—	A_1, A_2	A_2, A_5	—	A_1, A_2	A_2	—	A_1, A_3	A_5
S_4	A_2, A_3	—	A_1, A_5	—	A_1	—	A_1	A_2, A_3	A_5
S_5	A_3	—	A_1, A_2, A_5	—	A_1	A_2	A_1	—	A_5
S_6	A_2, A_3	A_1, A_2	A_5	—	A_1, A_2	—	—	A_1, A_2, A_3	A_5
S_7	A_3	A_2	A_1, A_5	—	A_1, A_2	—	A_1	—	A_5
S_8	—	A_1, A_2	A_1, A_5	A_1	A_1, A_2	—	—	A_1, A_3	A_5
S_9	—	A_1, A_2	A_1, A_2, A_5	—	A_1, A_2	A_2	A_1	A_1, A_3	A_5
S_{10}	—	A_1, A_2	A_1, A_5	A_1	A_1, A_2	—	A_1	A_1, A_3	A_1, A_5
S_{11}	A_2, A_3	—	A_1, A_2, A_5	—	A_1	A_2	A_1	A_2, A_3	A_5
S_{12}	A_2, A_3	A_1, A_2	A_1, A_5	—	A_1, A_2	—	A_1	A_1, A_2, A_3	A_5
S_{13}	A_2, A_3	A_1, A_2	A_2, A_5	—	A_1, A_2	A_2	—	A_1, A_2, A_3	A_5
S_{14}	A_3	A_2	A_1, A_2, A_5	—	A_1, A_2	A_2	A_1	—	A_5
S_{15}	A_3	A_2	A_1, A_5	A_1	A_1, A_2	—	A_1	—	A_1, A_5
S_{16}	A_2, A_3	A_1, A_2	A_1, A_2, A_5	—	A_1, A_2	A_2	A_1	A_1, A_2, A_3	A_5
S_{17}	A_2, A_3	A_1, A_2	A_1, A_2, A_5	A_1	A_1, A_2	A_2	A_1	A_1, A_2, A_3	A_1, A_5

Table 4: The connections between the coefficients of the SS's and the OS's according to Table 3.

4. Conclusions

The problem of reconstructing an original system, in general, is very complex and is solved in present primarily using numerical methods. At the same time, an analytic approach may be effective in particular cases. This paper shows a relative rigor of this approach if applied to SR-class systems. The method can also be applied to other classes of systems if there are transformations that connect the standard and the original systems.

This approach allows not only to obtain new systems that make an alternative to the known ones, as shown with an example of the Rössler system, but also to establish a set of candidate systems used to choose a unique real original system. Naturally, to solve the latter problem, a use of one observable is not sufficient. One also needs additional information about the system or a possibility to interact with the system in order to obtain such information. If the obtained information permits, we can determine not only the structure of the sought original system but also relations between its coefficients.

Coefficient combinations for the SS's	OS																
	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}
A_1	×	×		×	×		×	×	×	×	×	×		×	×	×	×
A_2	×		×		×		×		×		×		×	×	×	×	×
A_3	×				×		×							×	×		
A_5	×	×	×	×	×	×	×	×	×		×	×	×	×		×	
A_1, A_2	×	×	×		×	×	×	×	×	×	×	×	×	×	×	×	×
A_1, A_3	×	×	×		×		×	×	×	×				×	×		
A_1, A_5	×	×		×	×		×	×	×	×	×	×		×	×	×	×
A_2, A_3	×			×	×	×	×				×	×	×	×	×	×	×
A_2, A_5	×		×		×		×		×		×		×	×		×	
A_3, A_5	×				×		×							×			
A_1, A_2, A_3	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
A_1, A_2, A_5	×	×	×		×	×	×	×	×	×	×	×	×	×	×	×	×
A_1, A_3, A_5	×	×	×		×		×	×	×	×				×	×		
A_2, A_3, A_5	×			×	×	×	×				×	×	×	×		×	
A_1, A_2, A_3, A_5	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×

Table 5: Combinations of coefficients of the SS, which can change while the OS undergoes external effects.

Appendix A.

To prove Proposition 1, let us find a relation between the variables in systems (10) and (11).

Since $y_1 = x_2$, the second equation in (10) can be written as

$$\dot{y}_1 = b_0 + b_1 x_1 + b_2 y_1, \tag{A.1}$$

whence,

$$x_1 = \frac{\dot{y}_1 - b_2 y_1 - b_0}{b_1}. \tag{A.2}$$

By differentiating (A.1) with respect to t we get

$$\ddot{y}_1 = b_1 \dot{x}_1 + b_2 \dot{y}_1. \tag{A.3}$$

Substituting the expressions for $\dot{x}_1$ and $\dot{y}_1 = \dot{x}_2$ from (10) into (A.3) and making simplifications we get

$$\ddot{y}_1 = a_0 b_1 + b_0 b_2 + (a_1 b_1 + b_1 b_2) x_1 + (a_2 b_1 + b_2^2) x_2 + a_3 b_1 x_3. \tag{A.4}$$

Substitute now x_1 from (A.2) into (A.4) to find

$$x_3 = \frac{\ddot{y}_1 - (a_1 + b_2)\dot{y}_1 + (a_1 b_2 - a_2 b_1) y_1 + a_1 b_0 - a_0 b_1}{a_3 b_1}. \tag{A.5}$$

Differentiating (A.4) with respect to t we get

$$\dddot{y}_1 = (a_1b_1 + b_1b_2)\dot{x}_1 + (a_2b_1 + b_2^2)\dot{x}_2 + a_3b_1\dot{x}_3.$$

After making necessary substitutions from (10) we find

$$\begin{aligned}\dddot{y}_1 =& (a_1b_1 + b_1b_2)(a_0 + a_1x_1 + a_2x_2 + a_3x_3) + (a_2b_1 + b_2^2)\dot{x}_2 \\ &+ a_3b_1(c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_1x_2 + c_5x_1x_3 + c_6x_2x_3).\end{aligned} \tag{A.6}$$

Replace x_1 and x_3 in (A.6) using (A.2) and (A.5), correspondingly, and $\dot{x}_2$ with $\dot{y}_1$. By grouping similar terms, we get

$$\dddot{y}_1 = A_0 + A_1y_1 + A_2\dot{y}_1 + A_3\ddot{y}_1 + A_4y_1^2 + A_5y_1\dot{y}_1 + A_6y_1\ddot{y}_1 + A_7\dot{y}_1^2 + A_8\dot{y}_1\ddot{y}_1,$$

where the coefficients $A_0, \ldots, A_8$ are given in system (12). Setting $\dot{y}_1 = y_2$ and $\ddot{y}_1 = y_3$ we get system (11). The proposition is proved.

Appendix B.

Comparing relations (14) and (16) we find that the expressions for A_0, A_4, A_5, A_6, A_7, A_8 in these cases coincide and those for A_1, A_2, A_3 differ. To find b_0 in system (16), we set $a_2^{13} = a_2^{15} = a_2$, $a_3^{13} = a_3^{15} = a_3$, $b_1^{13} = b_1^{15} = b_1$, $b_2^{13} = b_2^{15} = b_2$, $c_0^{13} = c_0^{15} = c_0$, $c_5^{13} = c_5^{15} = c_5$, where the upper indices indicate whether the coefficients enter system (13) or (15), respectively. The new coefficient b_0^{15} is found, for example, by equating the relations for A_3 in (14) and (16). Then $b_2^{13} + c_3^{13} = b_2^{15} - b_0^{15}c_5^{15}/b_1^{15}$. Since $b_2^{13} = b_2^{15}$, we have that $b_0^{15} = -c_3^{13}b_1/c_5$. It follows from (13) that the numerical value of the new coefficient will be $b_0^{15} = c$. Substituting the relation for b_0^{15} into the expression for A_1 and A_2 from (16) we get $A_1 = -a_2b_1c_3^{13}$ and $A_2 = a_2b_1 - b_2c_3^{13}$, that are similar to the expressions for A_1 and A_2 in (14). Consequently, numerical values of all the coefficients in the reconstructions of systems (13) and (15) coincide and, hence, system (16) meets the conditions in Proposition 2.

Appendix C.

Let us analyze expression (12) by comparing (11) with (18). Since $A_8 = 0$ in (18), we have

$$c_5 = 0 \tag{C.1}$$

in the OS by (12). Consequently, $A_7 = 0$, which is verified by the reconstruction (18). It also follows from (C.1) that $A_6 = c_6$. But $A_6 = 0$ in (18) and, consequently,

$$c_6 = 0. \tag{C.2}$$

Since $A_5 \neq 0$ but (C.1) and (C.2) hold, we have

$$a_3 \neq 0, \qquad c_4 \neq 0. \tag{C.3}$$

Because $A_4 = 0$, it follows from (C.1), (C.2), and (C.3) that

$$b_2 = 0. \tag{C.4}$$

Consequently,

$$b_1 \neq 0, \tag{C.5}$$

for, otherwise, the second equation in (10) splits from the other equations.

In view of (C.1), (C.2), and (C.4), system (10) becomes

$$\begin{cases} \dot{x}_1 = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3, \\ \dot{x}_2 = b_0 + b_1 x_1, \\ \dot{x}_3 = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1 x_2. \end{cases}$$

Let us find the coordinates x_{10}, x_{20}, x_{30} of the equilibrium position of the system by setting $\dot{x}_1, \dot{x}_2, \dot{x}_3$ to zero,

$$\begin{cases} a_0 + a_1 x_{10} + a_2 x_{20} + a_3 x_{30} = 0, \\ b_0 + b_1 x_{10} = 0, \\ c_0 + c_1 x_{10} + c_2 x_{20} + c_3 x_{30} + c_4 x_{10} x_{20} = 0. \end{cases} \tag{C.6}$$

Since the observable $y_1(t) = x_2(t)$ oscillates about zero, see Fig. 4, we have $x_{20} = 0$ in (C.6). Then the system becomes

$$\begin{cases} a_0 + a_1 x_{10} + a_3 x_{30} = 0, \\ b_0 + b_1 x_{10} = 0, \\ c_0 + c_1 x_{10} + c_3 x_{30} = 0. \end{cases} \tag{C.7}$$

The second equation in (C.7) gives

$$x_{10} = -\frac{b_0}{b_1}. \tag{C.8}$$

Substituting (C.8) into the first and the third equations in (C.7) we get

$$\begin{cases} x_{30} = \dfrac{a_1 b_0 / b_1 - a_0}{a_3}, \\ x_{30} = \dfrac{c_1 b_0 / b_1 - c_0}{c_3}. \end{cases} \tag{C.9}$$

By equating the right-hand sides of (C.9), we have

$$\frac{a_1 b_0 - a_0 b_1}{a_3} = \frac{c_1 b_0 - c_0 b_1}{c_3}. \tag{C.10}$$

We have assumed, see Section 3.4, that the values of the coefficients of the OS may change in an arbitrary way if the system undergoes different effects, but relation (C.10) must always hold. This permits us to make the following conclusions.

1. The expressions in the left- and the right-hand sides of (C.10) must be equal to zero. Otherwise, the relation could change if the coefficients entering this relation change. Consequently,
$$a_1b_0 - a_0b_1 = c_1b_0 - c_0b_1 = 0. \tag{C.11}$$
2. Moreover, (C.10) could fail to hold if the coefficients of the OS change in the case where some of the four terms in (C.10) or (C.11) are distinct from zero. Hence, the following must be true:
$$a_1b_0 = a_0b_1 = c_1b_0 = c_0b_1 = 0. \tag{C.12}$$

Note that $A_0 = 0$ in system (19) as follows from (C.12), which is needed for the SS to be of the form (18).

For (C.12) (and (C.11)) to hold, it is sufficient that each of the monomials would have at least one coefficient zero. By (C.5), this implies that

$$a_0 = c_0 = 0. \tag{C.13}$$

Moreover, at least one of the two following conditions must be satisfied:

$$b_0 = 0 \tag{C.14}$$

or

$$a_1 = c_1 = 0. \tag{C.15}$$

So, by now we have that all candidate systems must have $a_0 = b_2 = c_0 = c_5 = c_6 = 0$ and $a_3 \neq 0$, $b_1 \neq 0$, $c_4 \neq 0$. The other coefficients that enter the right-hand sides of equations (19), $a_1, a_2, b_0, c_1, c_2, c_3$ may or may not be equal to zero in different versions of the OS.

Using (C.11) we get from (C.9) and (C.10) that

$$x_{30} = 0. \tag{C.16}$$

Substituting (C.13) and (C.16) into (C.7) we get

$$\begin{cases} a_1x_{10} = 0, \\ b_0 + b_1x_{10} = 0, \\ c_1x_{10} = 0. \end{cases} \tag{C.17}$$

If (C.14) holds, since $b_1 \neq 0$, the second equation in (C.17) gives that $x_{10} = 0$. If (C.15) holds and $b_0 \neq 0$, then $x_{10} = -b_0/b_1$.

Let us analyze the cases of (C.14) and (C.15).

1. If $b_0 = 0$, then $a_0 = b_0 = b_2 = c_0 = c_5 = c_6 = 0$ and the candidate systems can contain 8 nonzero coefficients $a_1, a_2, a_3, b_1, c_1, c_2, c_3, c_4$ where a_3, b_1, c_4 are always nonzero. Since the candidate systems have 6 nonzero coefficients in the right-hand sides, all possible combinations of 3 more coefficients out of the remaining 5 coefficients a_1, a_2, c_1, c_2, c_3

should be considered in view of Proposition 2. If two of them are zero, then the remaining three coefficients, together with a_3, b_1, c_4, will determine the structure of the candidate system. If we substitute the zero coefficients into system (19), we obtain one of the systems $S_1 - S_7$ given in Table 2. A system with $a_1 = a_3 = 0$ has been excluded from possible OS's, since $A_3 = 0$ in this case, which contradicts the SS (18). Systems with $a_2 = c_2 = 0$ or with $c_2 = c_3 = 0$ should also be disregarded, since that would give $A_1 = 0$.

2. If $b_0 \neq 0$ and $a_1 = c_1 = 0$, we get $a_0 = a_1 = b_2 = c_0 = c_1 = c_5 = c_6 = 0$, $a_3 \neq 0, b_0 \neq 0, b_1 \neq 0, c_4 \neq 0$. Since the OS must have 6 nonzero coefficients, the remaining two must be chosen from a_2, c_2, c_3. Set one of them to be zero each time and enter the others into the OS. It turns out that for $a_2 = 0$, $A_2 = 0$ by (19), and if $c_3 = 0$, then $A_3 = 0$, which contradicts the form of the SS (18). If $c_2 = 0$, the structure of the SS (18) does not change, so a system with nonzero $a_2, a_3, b_0, b_1, c_3, c_4$ (system S_8 in the Table 2) can be regarded as a candidate system.

Appendix D.

Let us consider an example of reconstruction of an SR-class OS from one observable in the case where the SS is of form (18). It follows from the analysis carried out in Appendix C that the only nonzero coefficients the OS can have are a_1, a_2, a_3, b_0, b_1, c_1, c_2, c_3, c_4 satisfying relations (19). Its general form is given in (24), and all possible choices for the OS are given in Table 2.

Assume we know that a change of the variable x_1 in the sought system does not depend on the value of the variable x_1. In other words, the coefficient a_1 in system (24) is zero. We will consider this fact as the first feature for choosing the OS from all candidate systems. As Table 4 shows, systems S_1, S_2, S_3, S_8, S_9, S_{10} have this property.

Assume that measurements of the observable were taken varying the conditions of the experiment and value of the only coefficient A_2 was changed in reconstruction (18), while the other coefficients A_1, A_3, A_5 remained unchanged. We then can use this as a second feature to single out the sought OS from other candidates. Table 5 shows that the second feature is enjoyed by the systems S_1, S_3, S_5, S_7, S_9, S_{11}, S_{13}, S_{14}, S_{15}, S_{16}, S_{17}. To make the analysis easier, we summarize this information in Table D.6.

OS		S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}
Features	1	×	×	×					×	×	×							
	2	×		×		×		×		×		×		×	×	×	×	×

Table D.6: Distribution of features 1 and 2 in candidate systems.

As the table shows, only systems S_1, S_3, and S_9 possess features 1 and 2. Hence, the number of systems to consider is reduced from 17 to 3. System S_1 is excluded from the considerations, since it has the same structure as the SS, for otherwise the problem could

be considered as solved after obtaining the reconstruction (18). Systems S_3 and S_9 have, respectively, 6 and 7 nonzero coefficients in the right-hand side, with the coefficients a_2, a_3, b_1, c_1, c_3, and c_4 entering both systems, whereas the coefficient c_2 is nonzero only in system S_9. If the information at hand does not allow to uniquely choose one of the systems, then it seems reasonable to choose a simpler system, which is S_3 in this case. Let us remark that this system, which is the same as system (17), was considered as an example of reconstruction in Section 3.2.

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